Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 654–669
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Gromov–Witten invariants of Fano hypersurfaces, revisited

Hironori Sakai

Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

Received 15 October 2007; received in revised form 21 December 2007; accepted 21 January 2008
Available online 10 February 2008

Abstract

The goal of this paper is to give an efficient computation of the genus zero three-point Gromov–Witten invariants of Fano hypersurfaces, starting from the Picard–Fuchs equation. This simplifies and to some extent explains the original computations of Jinzenji. The method involves solving a gauge-theoretic differential equation, and our main result is that this equation has a unique solution.

MSC: 53D45

Keywords: Quantum cohomology; $\mathcal{D}$-module; Birkhoff factorization

1. Introduction

Gromov–Witten invariants compute "numbers of pseudo-holomorphic curves" in a symplectic manifold. They are rigorously defined as integrals on moduli spaces of stable maps. Therefore it is difficult to calculate Gromov–Witten invariants directly from the definition.

An alternative method of computation comes from mirror symmetry. Although the mirror symmetry phenomenon is not yet completely understood, it suggests that Gromov–Witten invariants can be computed in terms of coefficients of power series solutions of certain differential equations. The most well known example is the quintic hypersurface in $\mathbb{C}P^4$; this is a Calabi–Yau 3-fold. Fano hypersurfaces are more elementary from the point of view of Gromov–Witten invariants, and it was established by Givental that certain Gromov–Witten invariants in this case are determined by the "Picard–Fuchs equation". The Picard–Fuchs equation of the quintic hypersurfaces in $\mathbb{C}P^4$ is $\left(\partial^4 - 5e^t(5\partial+4)(5\partial+3)(5\partial+2)(5\partial+1)\right)\psi(t) = 0$.

A hypersurface M_N^k of degree k in $\mathbb{C}P^{N-1}$ is Fano if and only if $N > k$, and the Picard–Fuchs equation is

$$\left(\partial^{N-1} - ke^t(k\partial+(k-1))\dots(k\partial+2)(k\partial+1)\right)\psi(t) = 0.$$

Before Givental's work, partial results on the quantum cohomology of Fano hypersurfaces had been obtained by Collino and Jinzenji [4] and Beauville [2]. Subsequently, Jinzenji [8] observed that a simple ansatz leads to the correct Gromov–Witten invariants and he obtained complicated but explicit formulae from this ansatz.

E-mail address: crescent@blueskyproject.net.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2008.01.008

The aim of this paper is to calculate the three-point Gromov–Witten invariants of a Fano hypersurface by using the method of [1,7] primarily when $N - k \geq 2$ (which we assume unless stated otherwise). In this method, the flat connection associated with the $\mathcal{D}$-module $\mathcal{D}/(\mathrm{PF})$ is "normalized" by applying the Birkhoff factorization. We shall show (as a consequence of Givental's work) that this method produces the correct three-point Gromov–Witten invariants.

The algorithm for the computation of three-point Gromov–Witten invariants from the quantum differential equations was introduced in [7], and applied to flag manifolds in [1], and our treatment of hypersurfaces is broadly similar. However, there are some special features in this case which make a separate discussion worthwhile. First, the differential equations in this case are o.d.e., rather than p.d.e., hence the integrability condition plays no role. Second, the o.d.e. which appears in the Birkhoff factorization can be integrated very explicitly, and this leads to purely algebraic formulae (whereas the algorithm in [1] required the solution of large systems of p.d.e.).

Computationally, our method is similar to Jinzenji's method, but considerably simpler. In Section 2 we review the cohomology algebra of hypersurfaces of the complex projective spaces. In Section 3 we discuss the Gromov–Witten invariants and the Dubrovin connection. The quantum differential system and Jinzenji's method are discussed in Section 4. In Section 5, we explain the loop group method and we compute a flat connection from a $\mathcal{D}$-module which is related to the quantum differential system. In Section 6, we discuss relations between families of connection 1-forms and $\mathcal{D}$-modules. The "adapted" gauge group is the most important object. In Section 7, we explain Jinzenji's results from our viewpoint and prove that our results agree with Jinzenji's results. We also prove that our results produce the Gromov–Witten invariants.

2. Hypersurfaces in the complex projective spaces

If we consider the hyperplane

$$H = \left\{[z_0, \ldots, z_{N-1}] \in \mathbb{C}P^{N-1} \mid z_0 = 0\right\}$$

of $\mathbb{C}P^{N-1}$ as a smooth divisor, the line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^{N-1}$ can be obtained from the divisor in the general theory of complex geometry. The line bundle $\mathcal{O}(1)$ is called the hyperplane bundle. The first Chern class b of the hyperplane bundle $\mathcal{O}(1)$ generates the cohomology algebra $H^*(\mathbb{C}P^{N-1}; \mathbb{C})$ of $\mathbb{C}P^{N-1}$. The tensor product of k copies of the line bundle $\mathcal{O}(1)$ is denoted by $\mathcal{O}(k)$. The zero locus of a holomorphic section of the line bundle $\mathcal{O}(k)$ is called a hypersurface of degree k in $\mathbb{C}P^{N-1}$ and the zero locus is denoted by M_N^k.

The $\mathbb{C}$-linear space $H^\sharp(M_N^k)$ of all pullbacks of cohomology classes via the inclusion map $\iota : M_N^k \to \mathbb{C}P^{N-1}$ is a subalgebra of the cohomology algebra of M_N^k. The subalgebra $H^\sharp(M_N^k)$ is generated by the pullback of the cohomology class b. The pullback is also denoted by b. Let b_i $(i = 1, \ldots, N-2)$ be a cup product of i copies of b and $b_0 = 1$. The vectors $b_0, \ldots, b_{N-2}$ form a $\mathbb{C}$-basis of the subalgebra $H^\sharp(M_N^k)$.

We assume that $N \geq 5$. Under this assumption, the Lefschetz theorem implies that the homomorphism $\iota_* : H_2(M_N^k; \mathbb{Z}) \to H_2(\mathbb{C}P^{N-1}; \mathbb{Z}) = \mathbb{Z}[\mathbb{C}P^1]$ induced by the inclusion is an isomorphism. Taking the generator A of $H_2(M_N^k; \mathbb{Z})$ with $\iota_* A = [\mathbb{C}P^1]$, we identify the second homology group $H_2(M_N^k; \mathbb{Z})$ with $\mathbb{Z}$ via the isomorphism.

There are two nondegenerate bilinear pairings over $\mathbb{C}$; one is the Kronecker pairing (the usual pairing)

$$\langle\ ,\ \rangle : H^2(M_N^k; \mathbb{C}) \otimes H_2(M_N^k; \mathbb{C}) \to \mathbb{C}; \quad \langle x, d\rangle = \int_d x$$

and the other is the Poincaré pairing

$$(\ ,\ ) : H^\sharp(M_N^k) \otimes H^\sharp(M_N^k) \to \mathbb{C}; \quad (x, y) = \int_{M_N^k} x \cup y.$$

Note that $g_{\mu\nu} := (b_\mu, b_\nu) = k\delta_{N-2}^{\mu+\nu}$.

3. Gromov–Witten invariants

The subalgebra $H^\sharp(M_N^k)$ is a Frobenius algebra in the following sense:

- The subalgebra $H^\sharp(M_N^k)$ is a commutative associative $\mathbb{C}$-algebra with unit 1.
- The pairing (,) is nondegenerate.
- For all $x, y, z \in H^\sharp(M_N^k)$, $(x \cup y, z) = (x, y \cup z)$.

Deforming the multiplicative structure on $H^\sharp(M_N^k)$ by a parameter t, we quantize the Frobenius algebra. To do this, we need the Gromov–Witten invariants.

We identify the unit 2-sphere S^2 in $\mathbb{R}^3$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ via the stereographic projection from the north pole

$$S^2 \to \mathbb{C} \cup \{\infty\}; \quad (x_1, x_2, x_3) \mapsto \frac{x_1 + \mathrm{i}x_2}{1 - x_3}.$$

Then S^2 is a Kähler manifold. If we take a homology class $d \in H_2(M_N^k; \mathbb{Z})$ and three cohomology classes $x_0, x_1, x_\infty \in H^\sharp(M_N^k)$ which are the Poincaré duals of submanifolds X_0, X_1, X_∞ of M_N^k respectively, then the (three-pointed genus zero) Gromov–Witten invariant $\mathrm{GW}_d(x_0, x_1, x_\infty)$ is roughly the number of holomorphic maps $u : S^2 \to M_N^k$ which have the following properties:

$$u(x_i) \in X_i \quad \text{for } i = 0, 1, \infty \in S^2 \quad \text{and} \quad u_*[S^2] = d.$$

For the rigorous definition, we need to study the moduli space of stable maps [5,9,11]. The Gromov–Witten invariants have the following properties.

Linearity axiom. For any $d \in H_2(M_N^k; \mathbb{Z})$, the map $\mathrm{GW}_d : H^\sharp(M_N^k)^{\otimes 3} \to \mathbb{C}$ is linear in each variable.
Effectivity axiom. If $d < 0$, $\mathrm{GW}_d = 0$.
Grading axiom. Let x_1, x_2, x_3 be homogeneous cohomology classes. Then $\mathrm{GW}_d(x_1, x_2, x_3) = 0$ unless

$$\sum_{i=1}^{3} \deg x_i = \dim M_N^k + 2\langle c_1(M_N^k), d\rangle = 2(N-2) + 2(N-k)d.$$

Symmetry axiom. Any permutation σ of $\{1, 2, 3\}$ preserves the Gromov–Witten invariant:

$$\mathrm{GW}_d(x_1, x_2, x_3) = \mathrm{GW}_d(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

Zero axiom. For any $x_1, x_2, x_3 \in H^\sharp(M_N^k)$,

$$\mathrm{GW}_0(x_1, x_2, x_3) = \int_{M_N^k} x_1 \cup x_2 \cup x_3.$$

Define the quantum product $*_t$ on M_N^k by

$$(x *_t y, z) = \sum_d \mathrm{GW}_d(x, y, z) e^{dt} \quad \text{for any } z \in H^\sharp(M_N^k).$$

Here $t \in \mathbb{C}$ is a parameter. Since the degree axiom implies that the sum is finite when $N - k \geq 1$, the quantum product is well defined. According to the theory of the gravitational descendents, the subspace $H^\sharp(M_N^k)$ is closed under the quantum product $*_t$ and forms a Frobenius algebra with the Poincaré pairing for every $t \in \mathbb{C}$ (see [10] Section 1). The quantum product makes the product bundle over $H^2(M_N^k; \mathbb{C})$ with fibre $H^\sharp(M_N^k)$ an algebra bundle. Namely the fibre $H^\sharp(M_N^k)$ on $tb \in H^2(M_N^k; \mathbb{C})$ with the product $*_t$ and the Poincaré pairing is a Frobenius algebra.

If we denote by q the exponential function e^t, then the quantum product $x *_t y$ can be regarded as the cohomology class with coefficients in polynomial $\mathbb{C}[q]$:

$$x *_t y = \sum_{\mu,\nu} \sum_d \mathrm{GW}_d(x, y, b_\mu) g^{\mu\nu} b_\nu q^d,$$

where $(g^{\mu\nu})$ is the inverse matrix of $(g_{\mu\nu})$. Therefore we think of the quantum product $*_t$ as a product "$\circ$" of the space $H^\sharp(M_N^k) \otimes \mathbb{C}[q]$. The algebra $\left(H^\sharp(M_N^k) \otimes \mathbb{C}[q], \circ\right)$ is called the small quantum cohomology and denoted by $\mathrm{QH}^\sharp(M_N^k)$.

The multiplication structure of the algebra bundle over $H^2(M_N^k;\mathbb{C})$ with fibre $H^\sharp(M_N^k)$ can be regarded as properties of a connection on the bundle which is called the Dubrovin connection. The Dubrovin connection is defined by

$$\nabla_{\mathrm{D}}^h := d + \frac{1}{h}(b *_t)\mathrm{d}t.$$

Here $h \in \mathbb{C}^\times$ is a parameter. If we think of the class b_ν as a constant section of the bundle, then $b_0, \ldots, b_{N-2}$ form a global frame. The connection 1-form of the Dubrovin connection with respect to the frame is given by

$$\Omega_{\mathrm{D}}^h = \frac{1}{h}\begin{pmatrix} \sum\limits_{\nu,d} \mathrm{GW}_d(b, b_0, b_\nu) g^{\nu,0} e^{dt} & \cdots & \sum\limits_{\nu,d} \mathrm{GW}_d(b, b_{N-2}, b_\nu) g^{\nu,0} e^{dt} \\ \vdots & & \vdots \\ \sum\limits_{\nu,d} \mathrm{GW}_d(b, b_0, b_\nu) g^{\nu,N-2} e^{dt} & \cdots & \sum\limits_{\nu,d} \mathrm{GW}_d(b, b_{N-2}, b_\nu) g^{\nu,N-2} e^{dt} \end{pmatrix} dt.$$

Note that the associativity of the quantum product implies the flatness of the Dubrovin connection.

4. The quantum differential system

Let ∇^* be the dual connection of the Dubrovin connection ∇_{D}^h. The system $\nabla^*\Psi = 0$ of differential equations for Ψ is called the quantum differential system. The connection form of the dual connection ∇^* is $-(\Omega_{\mathrm{D}}^h)^{\mathrm{T}}$ with respect to the dual basis $b^0, \ldots, b^{N-2}$ of the basis $b_0, \ldots, b_{N-2}$ for the Poincaré pairing. Putting $\Psi = \sum_{m=0}^{N-2} \psi_{N-2-m} b^{N-2-m}$, the quantum differential equation is explicitly given by

$$\frac{\partial \psi_{N-2-m}}{\partial t} = \psi_{N-1-m}(t) + \sum_{d=1}^{\infty} L_m^d e^{dt} \psi_{N-1-m-(N-k)d}(t) \quad (m = 1, \ldots, N-2),$$
$$\frac{\partial \psi_{N-2}}{\partial t} = \sum_{d=1}^{\infty} L_0^d e^{dt} \psi_{N-1-(N-k)d}(t),$$

where $L_m^d = k^{-1}\,\mathrm{GW}_d(b, b_{N-2-m}, b_{m-1+(N-k)d})$.

The following important fact was proved by Givental in [6].

Theorem 4.1 (*Givental*). *Assume that $N - k \geq 2$. The Gauss–Manin system is equivalent to the Picard–Fuchs equation:*

$$\left(\partial^{N-1} - ke^t(k\partial + (k-1)) \ldots (k\partial + 2)(k\partial + 1)\right)\psi_0(t) = 0,$$

where ∂ means $\frac{\partial}{\partial t}$.

Jinzenji proposed a method for completing the Gromov–Witten invariants of Fano hypersurfaces from the Picard–Fuchs equation in [8]. We explain briefly Jinzenji's method for the Fano hypersurface M_5^3.

The Gauss–Manin system for M_5^3 is

$$\frac{\partial \psi_0}{\partial t} = \psi_1(t),$$
$$\frac{\partial \psi_1}{\partial t} = \psi_2(t) + L_2^1 e^t \psi_0(t),$$
$$\frac{\partial \psi_2}{\partial t} = \psi_3(t) + L_1^1 e^t \psi_1(t),$$
$$\frac{\partial \psi_3}{\partial t} = +L_0^1 e^t \psi_2(t) + L_0^2 e^{2t} \psi_0(t).$$

Reducing this system, we have

$$\left(\partial^4 - e^t\left((L_2^1 + L_1^1 + L_0^1)\partial^2 + (2L_2^1 + L_1^1)\partial + L_2^1\right) - e^{2t}(L_2^2 - L_0^1 L_2^1)\right)\psi_0 = 0.$$

Givental's theorem implies that this differential equation becomes

$$\left(\partial^4 - 3e^t(3\partial+2)(3\partial+1)\right)\psi_0 = 0.$$

Thus we conclude that

$$L_0^1 = 6, \qquad L_1^1 = 15, \qquad L_2^1 = 6, \qquad L_0^2 = 36.$$

5. Birkhoff factorization

We modify the quantum differential system with the parameter h as follows:

$$h\frac{\partial\psi_{N-2-m}}{\partial t} = \psi_{N-1-m}(t) + \sum_{d\geq 1} L_m^d e^{dt}\psi_{N-1-m-(N-k)d}(t),$$
$$h\frac{\partial\psi_{N-2}}{\partial t} = \sum_{d\geq 1} L_0^d e^{dt}\psi_{N-1-(N-k)d}(t),$$

(where $m = 1,\ldots,N-2$). We shall use the following modification of Theorem 4.1:

Proposition 5.1 ([6] Corollary 9.2). *Assume that $N-k\geq 2$. The quantum differential system with the parameter h is equivalent to the Picard–Fuchs equation with parameter h:*

$$\left((h\partial)^{N-1} - kqh^{k-1}(k\partial+(k-1))\ldots(k\partial+2)(k\partial+1)\right)\psi_0(t) = 0,$$

where $q = e^t$.

In this section and the next section, we will show that the Picard–Fuchs equation with h gives the Gromov–Witten invariants by using the Birkhoff factorization, as in [7,1].

Let $\Lambda = \mathbb{C}[q]$ be the polynomial ring generated by q and $\mathcal{D}$ be the module generated by $h\partial$ over $\Lambda(h)$. First of all we consider the $\mathcal{D}$-module $\mathcal{M}^h = \mathcal{D}/(P^{N,k})$, where $(P^{N,k})$ is the left ideal generated by the operator

$$P^{N,k} = (h\partial)^{N-1} - kqh^{k-1}(k\partial+(k-1))\ldots(k\partial+2)(k\partial+1).$$

Second, we introduce a family of (flat) connection 1-forms $\Omega_{\mathrm{PF}}^h = \frac{1}{h}R^h(q)\mathrm{d}t$. Put $P_0 = 1$, $P_1 = h\partial,\ldots,P_{N-2} = (h\partial)^{N-2}$. Then the equivalence classes $[P_0],\ldots,[P_{N-2}]$ form a $\Lambda(h)$-basis of the $\mathcal{D}$-module $\mathcal{M}^h$. We define $R^h(q)$ by

$$h\partial([P_0],\ldots,[P_{N-2}]) = ([P_0],\ldots,[P_{N-2}])R^h(q), \quad \text{i.e.} \quad h\partial[P_\alpha] = \sum_{\beta=0}^{N-2}(R^h(q))_\alpha^\beta[P_\beta].$$

Then $\Omega_{\mathrm{PF}}^h = \frac{1}{h}R^h(q)\mathrm{d}t$ is of the form

$$\Omega_{\mathrm{PF}}^h = \frac{1}{h}\omega + \theta_0 + h\theta_1 + \cdots + h^{k-2}\theta_{k-2},$$

where $\omega,\theta_0,\ldots,\theta_p$ are matrix-valued 1-forms independent of h.

Finally we obtain a connection from Ω_{PF}^h by using the Birkhoff decomposition which is a candidate Dubrovin connection. We consider h as a parameter in $S^1\subset\mathbb{C}$. Since Ω_{PF}^h is flat, there is a map L from an open subset V of $\mathbb{C}$ to the loop group $\Lambda\mathrm{GL}(\mathbb{C}^{N-1})$ such that $\Omega_{\mathrm{PF}}^h = L^{-1}dL$. The loop group $\Lambda\mathrm{GL}(\mathbb{C}^{N-1})$ is the group of all smooth map from S^1 to $\mathrm{GL}(\mathbb{C}^{N-1})$.

Let $L = L_-L_+$ be the Birkhoff decomposition of L, where L_- extends holomorphically to $1<|h|\leq\infty$ and L_+ to $|h|<1$, and $L_-|_{h=0} = I$. In other words, L_- and L_+ have expansions in h as follows:

$$L_- = I + \frac{1}{h}A_1 + \frac{1}{h}A_2 + \cdots, \quad L_+ = Q_0(I + hQ_1 + h^2Q_2 + \cdots).$$

Note that the Birkhoff decomposition exists if and only if L takes values in the big cell of the loop group. Since the big cell is an open dense subset of the loop group, we can choose γ such that $\gamma L(q)$ belongs to the big cell for all q in a sufficiently small set V.

If we expand $\hat{\Omega}^h = (L_-)^{-1} dL_-$ as a series in h, then only negative powers of h appear. On the other hand, we have

$$\begin{aligned}(L_-)^{-1} dL_- &= (LL_+^{-1})^{-1} d(LL_+^{-1}) \\ &= (L_+ L^{-1}) \left((dL) L_+^{-1} + L d(L_+^{-1}) \right) \\ &= L_+ (L^{-1} dL) L_+^{-1} + L_+ d(L_+^{-1}) \\ &= L_+ \left(\frac{1}{h}\omega + \theta_0 + \cdots + h^p \theta_p \right) L_+ + L_+ d(L_+^{-1}).\end{aligned}$$

Since the negative powers of h disappear except for $\frac{1}{h} Q_0 \omega Q_0^{-1}$ in the above expression, we conclude that $\hat{\Omega}^h = \frac{1}{h} Q_0 \omega Q_0^{-1}$. In the next section we will see that $\hat{\nabla}^h = d + \hat{\Omega}^h$ agrees with the restricted Dubrovin connection ∇_{D}^h, where the restricted Dubrovin connection is the restriction of the Dubrovin connection to the trivial bundle $H^\sharp(M_N^k) \times H^2(M_N^k; \mathbb{C}) \to H^\sharp(M_N^k)$. The restriction is well defined because the quantum product on $H^\sharp(M_N^k)$ is closed (see Section 3).

It is difficult to execute the Birkhoff decomposition in general. Note that we need only L_+ to work out $\hat{\Omega}^h$. Since the non-negative powers of h in $(L_-)^{-1} dL_-$ disappear, we can obtain differential equations for L_+:

Proposition 5.2 (*[1]*). $L_+ = Q_0(I + hQ_1 + h^2 Q_2 + \cdots)$ *satisfies the following differential equations:*

$(\mathcal{L}_0)$ $dQ_0 = Q_0(\theta_0 + [Q_1, \omega])$,

$(\mathcal{L}_1)$ $dQ_1 = \theta_1 + [Q_1, \theta_0] + [Q_2, \omega] - [Q_1, \omega] Q_1$, *and*

$(\mathcal{L}_i)$ $dQ_i = \theta_i + Q_1 \theta_{i-1} + \cdots + Q_{i-1}\theta_1 + [Q_i, \theta_0] + [Q_{i+1}, \omega] - [Q_1, \omega] Q_i$

for $i \geq 2$.

Here, $\mathcal{L}_i$ denotes the equation for Q_i.

To calculate L_+ we introduce some notation. Let $E_{i,j}$ be the $N-1 \times N-1$ matrix with (i, j)-component 1 and all other components zero.

For an integer n with $|n| \leq N-1$, we define an $N-1 \times N-1$ matrix $\mathrm{diag}_n(a_1, \ldots, a_{N-2-|n|})$ by

$$\mathrm{diag}_n(a_1, \ldots, a_{N-2-|n|}) = \begin{cases} \displaystyle\sum_{i=1}^{N-2-n} a_i E_{i,n+i} & (n \geq 0), \\ \displaystyle\sum_{i=1}^{N-2+n} a_i E_{i-n,i} & (n < 0). \end{cases}$$

We call this an n-diagonal matrix. The identity $E_{i,j} E_{\alpha,\beta} = \delta_{j,\alpha} E_{i,\beta}$ implies that the product of an n-diagonal matrix and an m-diagonal matrix is an $(n+m)$-diagonal matrix.

The matrix $\mathrm{diag}_{-1}(1, \ldots, 1)$ is denoted by I_{-1}. For a matrix $A = (a_{i,j})$ and non-negative integer n, we call the matrix $\mathrm{diag}_n(a_{1,1+n}, a_{2,2+n}, \ldots, a_{N-2-n,N-2})$ the n-diagonal component of A.

Furthermore we define non-negative integers λ_i^k as in the previous section by

$$k \prod_{j=1}^{k-1} (kX + j) = \lambda_{k-1}^k X^{k-1} + \lambda_{k-2}^k X^{k-2} + \cdots + \lambda_1^k X + \lambda_0^k.$$

The Picard–Fuchs operator of M_N^k is described in terms of λ_i^k as follows:

$$P^{N,k} = (h\partial)^{N-1} - q \left(\lambda_{k-1}^k (h\partial)^{k-1} + \lambda_{k-2}^k h (h\partial)^{k-2} + \cdots + \lambda_0^k h^{k-1} \right).$$

If there is no danger of confusion, we omit the upper suffix of λ_i^k.

Recall that $P_i = (h\partial)^i$. We have $[\partial P_0] = \frac{1}{h}[P_1], \ldots, [\partial P_{N-3}] = \frac{1}{h}[P_{N-2}]$ and

$$[\partial P_{N-2}] = \frac{1}{h}[(h\partial)^{N-1}] = \frac{1}{h}\lambda_{k-1}q[P_{k-1}] + \lambda_{k-2}q[P_{k-2}] + \cdots + h^{k-2}\lambda_0 q[P_0].$$

Thus

$$\Omega^h_{\mathrm{PF}} = \frac{1}{h}\omega + \theta_0 + h\theta_1 + \cdots + h^{k-2}\theta_{k-2},$$

where

$$\omega = (I_{-1} + qR_{-1})\mathrm{d}t = I_{-1}dt + R_{-1}dq,$$
$$\theta_0 = R_0 dq,\ \ \ldots, \theta_{k-2} = R_{k-2}dq,$$
$$R_i = \mathrm{diag}_{N-k+i}(0, \ldots, 0, \lambda_{k-2-i}).$$

Note that $qdt = dq$ because $q = e^t$.

The following properties will be useful in the calculation of L_+.

Proposition 5.3 ([7]). *If we set $\deg h = 2$ and $\deg q = 2(N-k)$, the following statements hold.*

(i) *If the (α, β)-component of L_+ does not vanish, it has degree $2(\beta - \alpha)$.*
(ii) *If the (α, β)-component of Q_i does not vanish, it has degree $2(\beta - \alpha - i)$.*
(iii) *There is a matrix X such that $Q_0 = \exp X$ and the n-diagonal component of X vanishes for $n \le 1$.*
(iv) *For $i \ge 1$ and $n \le i+1$, the n-diagonal component of Q_i vanishes.*

According to the above proposition, we may assume that the Q_i are of the form

$$Q_0 = I + qQ_0^1 + q^2Q_0^2 + \cdots = I + \sum_{\alpha\ge 1} q^\alpha Q_0^\alpha,$$
$$Q_i = \sum_{\alpha\ge 1} q^\alpha Q_i^\alpha \quad (i \ge 1),$$

where Q_i^α is a constant $(i + \alpha(N-k))$-diagonal matrix. Thus Q_i^α vanishes if α is greater than $(N-2-i)/(N-k)$.

Before solving the equations for L_+, we note the following identities:

(i) $R_j Q_i^\alpha = 0 (i, \alpha \ge 0,\ j \ge -1)$.
(ii) $[Q_1, \omega] = [Q_1^1, I_{-1}]dq + \sum_{\alpha\ge 1} q^\alpha \left([Q_1^{\alpha+1}, I_{-1}] + Q_1^\alpha R_{-1}\right) dq.$

First, we consider the equation $(\mathcal{L}_0)$. The left hand side is $dQ_0 = \sum_{\alpha\ge 1}\alpha q^{\alpha-1}Q_0^\alpha$, while the right hand side is

$$\begin{aligned} Q_0(\theta_0 + [Q_1, \omega]) &= Q_0\left((R_0 + [Q_1^1, I_{-1}]) + \sum_{\beta\ge 1} q^\beta\left([Q_1^{\beta+1}, I_{-1}] + Q_1^\beta R_{-1}\right)\right) dq \\ &= (R_0 + [Q_1^1, I_{-1}])dq + q\left(Q_0^1(R_0 + [Q_1^1, I_{-1}]) + [Q_1^2, I_{-1}] + Q_1^1 R_{-1}\right) dq \\ &\quad + \sum_{\gamma\ge 2} q^\gamma \left([Q_1^{\gamma+1}, I_{-1}] + Q_1^\gamma R_{-1} + Q_0^\gamma(R_0 + [Q_1^1, I_{-1}])\right. \\ &\quad \left. + \sum_{\alpha+\beta=\gamma} Q_0^\alpha([Q_1^{\beta+1}, I_{-1}] + Q_1^\beta R_{-1})\right) dq. \end{aligned}$$

Thus we have

$$\begin{aligned} Q_0^1 &= R_0 + [Q_1^1, I_{-1}], \\ 2Q_0^2 &= Q_0^1(R_0 + [Q_1^1, I_{-1}]) + [Q_1^2, I_{-1}] + Q_1^1 R_{-1}, \\ \gamma Q_0^\gamma &= [Q_1^\gamma, I_{-1}] + Q_1^{\gamma-1}R_{-1} + Q_0^{\gamma-1}(R_0 + [Q_1^1, I_{-1}]) \\ &\quad + \sum_{\alpha,\beta\ge 1,\ \alpha+\beta=\gamma-1} Q_0^\alpha([Q_1^{\beta+1}, I_{-1}] + Q_1^\beta R_{-1}) \quad (\gamma \ge 3). \end{aligned}$$

Second, we consider the equation $(\mathcal{L}_1)$. The left hand side is $dQ_1 = \sum_{\alpha\geq 1} \alpha q^{\alpha-1} Q_1^\alpha dq$, while the terms in the right hand side are

$$\begin{aligned}
&\theta_1 = R_1 dq,\\
&[Q_1, \theta_0] = \sum_{\alpha\geq 1} q^\alpha Q_1^\alpha R_0 dq\\
&[Q_2, \omega] = [Q_2^1, I_{-1}]dq + \sum_{\alpha\geq 1} q^\alpha \left([Q_2^{\alpha+1}, I_{-1}] + Q_2^\alpha R_{-1}\right) dq,\\
&[Q_1, \omega]Q_1 = \sum_{\alpha\geq 1} q^\alpha [Q_1^1, I_{-1}] Q_1^\alpha dq + \sum_{\gamma\geq 2} q^\gamma \sum_{\alpha+\beta=\gamma} \left([Q_1^{\alpha+1}, I_{-1}] + Q_1^\alpha R_{-1}\right) Q_1^\beta dq.
\end{aligned}$$

Therefore

$$\begin{aligned}
Q_1^1 &= R_1 + [Q_2^1, I_{-1}],\\
2Q_1^2 &= Q_1^1 R_0 + [Q_2^2, I_{-1}] + Q_2^1 R_{-1} - [Q_1^1, I_{-1}] Q_1^1,\\
\gamma Q_1^\gamma \ &= \ Q_1^{\gamma-1} R_0 + [Q_2^\gamma, I_{-1}] + Q_2^{\gamma-1} R_{-1} - [Q_1^1, I_{-1}] Q_1^{\gamma-1}\\
&\quad - \sum_{\alpha,\beta\geq 1,\ \alpha+\beta=\gamma-1} \left([Q_1^{\alpha+1}, I_{-1}] + Q_1^\alpha R_{-1}\right) Q_1^\beta \quad (\gamma \geq 3).
\end{aligned}$$

Finally, we consider the equation $(\mathcal{L}_i)$. The right hand side is $dQ_i = \sum_{\alpha\geq i} \alpha q^{\alpha-1} Q_i^\alpha dq$, and the terms in the right hand side are

$$\begin{aligned}
&\theta_i = R_i dq,\\
&Q_j \theta_{i-j} = \sum_{\alpha\geq 1} q^\alpha Q_j^\alpha R_{i-j} dq,\\
&[Q_i, \theta_0] = \sum_{\alpha\geq 1} q^\alpha Q_i^\alpha R_0 dq,\\
&[Q_{i+1}, \omega] = [Q_{i+1}^1, I_{-1}]dq + \sum_{\alpha\geq 1} q^\alpha \left([Q_{i+1}^{\alpha+1}, I_{-1}] + Q_{i+1}^\alpha R_{-1}\right) dq,\\
&[Q_1, \omega]Q_i = \sum_{\alpha\geq 1} q^\alpha [Q_1^1, I_{-1}] Q_i^\alpha dq + \sum_{\gamma\geq 2} q^\gamma \sum_{\alpha+\beta=\gamma} \left([Q_1^{\alpha+1}, I_{-1}] + Q_1^\alpha R_{-1}\right) Q_i^\beta dq.
\end{aligned}$$

Thus

$$\begin{aligned}
Q_i^1 &= R_i + [Q_{i+1}^1, I_{-1}],\\
2Q_i^2 &= \sum_{j=1}^{i+1} Q_j^1 R_{i-j} + [Q_{i+1}^2, I_{-1}] - [Q_1^1, I_{-1}] Q_i^1,\\
\gamma Q_i^\gamma \ &= \ \sum_{j=1}^{i+1} Q_j^{\gamma-1} R_{i-j} + [Q_{i+1}^\gamma, I_{-1}] - [Q_1^1, I_{-1}] Q_i^{\gamma-1}\\
&\quad - \sum_{\alpha,\beta\geq 1,\ \alpha+\beta=\gamma-1} \left([Q_1^{\alpha+1}, I_{-1}] + Q_1^\alpha R_{-1}\right) Q_i^\beta \quad (\gamma \geq 3).
\end{aligned}$$

Looking at the above identities, we see that Q_i^γ is determined by the following information:

(i) $Q_i^\alpha \quad (\alpha > \gamma)$.
(ii) $Q_j^\beta \quad (j < i,\ 0 \leq \beta \leq k-2)$.

Since $Q_{k-2}^1 = R_{k-2} = \mathrm{diag}_{N-2}(\lambda_0) = \mathrm{diag}_{N-2}(k!)$, we can determine $L_+ = Q_0(I + hQ_1 + \cdots + h^{k-2} Q_{k-2})$ from Q_{k-2}^1 explicitly.

Example. We apply the above results for M_7^5. Its Picard–Fuchs operator is

$$P^{7,5} = (h\partial)^6 - 5qh^4(5\partial + 4)(5\partial + 3)(5\partial + 2)(5\partial + 1).$$

First we calculate Ω_{PF}^h:

$$\Omega_{\mathrm{PF}}^h = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 120qh^3 \\ 1/h & 0 & 0 & 0 & 0 & 1250qh^2 \\ 0 & 1/h & 0 & 0 & 0 & 4375qh \\ 0 & 0 & 1/h & 0 & 0 & 6250q \\ 0 & 0 & 0 & 1/h & 0 & 3125q/h \\ 0 & 0 & 0 & 0 & 1/h & 0 \end{pmatrix} \mathrm{d}t.$$

Thus we have

$$\begin{aligned} R_{-1} &= \mathrm{diag}_1(0, 0, 0, 0, 3125), \\ R_0 &= \mathrm{diag}_2(0, 0, 0, 6250), \\ R_1 &= \mathrm{diag}_3(0, 0, 4375), \\ R_2 &= \mathrm{diag}_4(0, 1250), \\ R_3 &= \mathrm{diag}_5(120). \end{aligned}$$

Second, we calculate Q_i and L_+. We can put

$$\begin{aligned} Q_0 &= I + qQ_0^1 + q^2 Q_0^2, \\ Q_1 &= qQ_1^1 + q^2 Q_1^2, \\ Q_2 &= qQ_2^1, \\ Q_3 &= qQ_3^1. \end{aligned}$$

where Q_i^α is a constant $(i + 2\alpha)$-diagonal matrix. We can determine them in the following order:

$$\begin{aligned} Q_3^1 &= R_3 = \mathrm{diag}_5(120), \\ Q_2^1 &= R_2 + [Q_3^1, I_{-1}] = \mathrm{diag}_4(120, 1130), \\ Q_1^1 &= R_1 + [Q_2^1, I_{-1}] = \mathrm{diag}_3(120, 1010, 3245), \\ Q_0^1 &= R_0 + [Q_1^1, I_{-1}] = \mathrm{diag}_2(120, 890, 2235, 3005), \\ Q_1^2 &= \frac{1}{2}\left([Q_1^1, R_0] + [Q_2^1, R_{-1}] - [Q_1^1, I_{-1}]Q_1^1\right) = \mathrm{diag}_5(367\,800), \\ Q_0^2 &= \frac{1}{2}\left(Q_0^1\left(R_0 + [Q_1^1, I_{-1}]\right) + [Q_1^1, R_{-1}] + [Q_1^2, I_{-1}]\right) = \mathrm{diag}_4(318\,000, 2\,731\,450). \end{aligned}$$

Thus we have

$$\begin{aligned} Q_0 &= I + qQ_0^1 + q^2 Q_0^2 \\ &= I + q\mathrm{diag}_2(120, 890, 2235, 3005) + q^2\mathrm{diag}_4(318\,000, 2\,731\,450) \\ &= \begin{pmatrix} 1 & 0 & 120q & 0 & 318\,000q^2 & 0 \\ 0 & 1 & 0 & 890q & 0 & 2\,731\,450q^2 \\ 0 & 0 & 1 & 0 & 2235q & 0 \\ 0 & 0 & 0 & 1 & 0 & 3005q \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$
\begin{aligned}
L_+ &= Q_0(I + hQ_1 + h^2Q_2 + h^3Q_3) \\
&= \begin{pmatrix}
1 & 0 & 120q & 120qh & 120qh^2 + 318\,000q^2 & 120qh^3 + 757\,200q^2h \\
0 & 1 & 0 & 890q & 1010qh & 1130qh^2 + 2\,731\,450q^2 \\
0 & 0 & 1 & 0 & 2235q & 3245qh \\
0 & 0 & 0 & 1 & 0 & 3005q \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\end{aligned}
$$

Finally, we calculate $\hat{\Omega}^h$:

$$
\begin{aligned}
\hat{\Omega}^h &= \frac{1}{h} Q_0 \omega (Q_0)^{-1} \\
&= \frac{1}{h} Q_0 (I_{-1} + qR_{-1})(Q_0)^{-1} dt \\
&= \frac{1}{h} \begin{pmatrix}
0 & 120q & 0 & 211\,200q^2 & 0 & 31\,320\,000q^3 \\
1 & 0 & 770q & 0 & 692\,500q^2 & 0 \\
0 & 1 & 0 & 1345q & 0 & 211\,200q^2 \\
0 & 0 & 1 & 0 & 770q & 0 \\
0 & 0 & 0 & 1 & 0 & 120q \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\end{aligned}
$$

We will see that $\hat{\Omega}^h$ agrees with the restricted Dubrovin connection Ω^h_{D}.

Example. We apply the above results for M_5^4.

$$P^{5,4} = (h\partial)^4 - 4qh^3(4\partial + 3)(4\partial + 2)(4\partial + 1).$$

First we calculate Ω^h_{PF}:

$$
\Omega^h_{\mathrm{PF}} = \begin{pmatrix}
0 & 0 & 0 & 24qh^2 \\
1/h & 0 & 0 & 176qh \\
0 & 1/h & 0 & 384q \\
0 & 0 & 1/h & 256q/h
\end{pmatrix} \mathrm{d}t.
$$

Thus we have

$$
\begin{aligned}
R_{-1} &= \mathrm{diag}_0(0, 0, 0, 256), \\
R_0 &= \mathrm{diag}_1(0, 0, 384), \\
R_1 &= \mathrm{diag}_2(0, 176), \\
R_2 &= \mathrm{diag}_3(24).
\end{aligned}
$$

Second, we calculate Q_i and L_+. We can put

$$
\begin{aligned}
Q_0 &= I + qQ_0^1 + q^2Q_0^2 + q^3Q_0^3, \\
Q_1 &= qQ_1^1 + q^2Q_1^2, \\
Q_2 &= qQ_2^1.
\end{aligned}
$$

where Q_i^α is a constant $(i + \alpha)$-diagonal matrix. We can determine them in the following order:

$$
\begin{aligned}
Q_2^1 &= R_2 = \mathrm{diag}_3(24), \\
Q_1^1 &= R_1 + [Q_2^1, I_{-1}] = \mathrm{diag}_2(24, 152), \\
Q_0^1 &= R_0 + [Q_1^1, I_{-1}] = \mathrm{diag}_1(24, 128, 232),
\end{aligned}
$$

$$
\begin{aligned}
Q_1^2 &= \frac{1}{2}\left([Q_1^1, R_0] + [Q_2^1, R_{-1}] - [Q_1^1, I_{-1}]Q_1^1\right) = \mathrm{diag}_3(5856),\\
Q_0^2 &= \frac{1}{2}\left(Q_0^1\left(R_0 + [Q_1^1, I_{-1}]\right) + [Q_1^2, I_{-1}] + Q_1^1 R_{-1}\right) = \mathrm{diag}_2(4464, 31\,376),\\
Q_0^3 &= \frac{1}{3}\left([Q_1^3, I_{-1}] + Q_1^2 R_{-1} + Q_0^2(R_0 + [Q_1^1, I_{-1}]) + Q_0^1([Q_1^2, I_{-1}] + Q_1^1 R_{-1})\right)\\
&= \mathrm{diag}_3(1\,109\,376).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
Q_0 &= I + qQ_0^1 + q^2 Q_0^2 + q^3 Q_0^3\\
&= I + q\,\mathrm{diag}_1(24, 128, 232) + q^2\mathrm{diag}_2(4464, 31\,376) + q^3\mathrm{diag}_3(1\,109\,376)\\
&= \begin{pmatrix} 1 & 24q & 4464q^2 & 1\,109\,376q^3\\ 0 & 1 & 128q & 31\,376q^2\\ 0 & 0 & 1 & 232q\\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
L_+ &= Q_0(I + hQ_1 + h^2 Q_2)\\
&= \begin{pmatrix} 1 & 24q & 24qh + 4464q^2 & 24qh^2 + 9504q^2h + 1\,109\,376q^3\\ 0 & 1 & 128q & 152qh + 31\,376q^2\\ 0 & 0 & 1 & 232q\\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Finally, we calculate $\hat{\Omega}^h$:

$$
\begin{aligned}
\hat{\Omega}^h &= \frac{1}{h} Q_0 \omega (Q_0)^{-1}\\
&= \frac{1}{h}\begin{pmatrix} 24q & 3888q^2 & 504\,576q^3 & 18\,323\,712q^4\\ 1 & 104q & 13\,600q^2 & 504\,576q^3\\ 0 & 1 & 104q & 3888q^2\\ 0 & 0 & 1 & 24q \end{pmatrix}.
\end{aligned}
$$

In fact, $\hat{\Omega}^h$ does not agree with the Dubrovin connection Ω_{D}^h. But we will see that the modified connection

$$
-\frac{4!q}{h} I dt + \hat{\Omega}^h = \frac{1}{h}\begin{pmatrix} 0 & 3888q^2 & 504\,576q^3 & 18\,323\,712q^4\\ 1 & 80q & 13\,600q^2 & 504\,576q^3\\ 0 & 1 & 80q & 3888q^2\\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

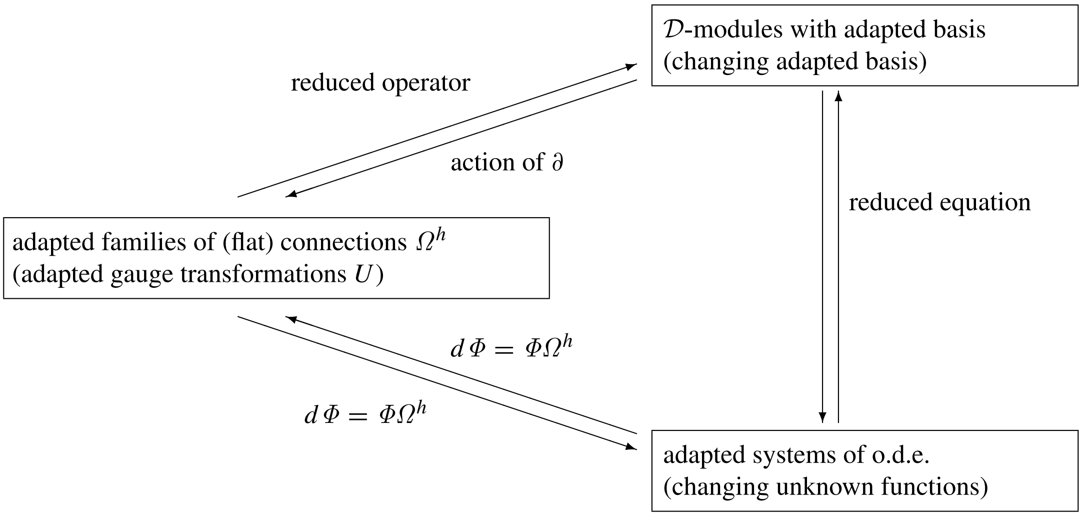
agrees with Ω_{D}^h.

The above algorithm can easily be implemented in Maple,[1] Mathematica etc. It is more elementary than the method of [8,3].

6. Adapted gauge group

There are three important ingredients of the theory, i.e., $\mathcal{D}$-modules (with adapted basis), "adapted" systems of differential equations, and "adapted" (flat) connections. These are closely related to each other.

[1] A Maple program can be found at http://sakai.blueskyproject.net/via_dmod/.



Let t be a coordinate function on $\mathbb{C}$ and $q = e^t$. We consider families of connections and the gauge group on the trivial bundle $\mathbb{C} \times \mathbb{C}^{N-1} \to \mathbb{C}$. The space of connection 1-forms on the bundle is the space of $\mathrm{End}(\mathbb{C}^{N-1})$-valued functions on $\mathbb{C}$ and the gauge group is the space of $\mathrm{GL}(\mathbb{C}^{N-1})$-valued functions $\mathbb{C}$. Therefore if we think of h as the loop parameter, then the space of families of gauge transformations is identified with functions on $\mathbb{C}$ with values in the loop group $\Lambda\mathrm{GL}(\mathbb{C}^{N-1})$.

A matrix-valued function A in q and h is called homogeneous if the n-diagonal component of A has degree $2n$ for each n. A $\Lambda\mathrm{GL}(\mathbb{C}^{N-1})$-valued function U on $\mathbb{C}$ is called adapted if U satisfies following properties.

(P) U is a $\mathrm{GL}(\mathbb{C}^{N-1})$-valued polynomial in q and h.
(H) U is homogeneous.
(I) $U|_{q=0} = I$. (initial condition)

Let U be a $\Lambda\mathrm{GL}(\mathbb{C}^{N-1})$-valued adapted function on $\mathbb{C}$. We can easily compute the inverse of U. The equation $UV = I$ is equivalent to $U_0V_0 = I$ and $\sum_{\alpha+\beta=\gamma} U_\alpha V_\beta = 0$ for $\gamma \geq 1$ if we put $V = \sum_{\alpha=0}^{N-2} V_\alpha$, where U_α and V_α are the α-diagonal components of U and V, respectively. The first identity and $U_0 = I$ imply $V_0 = I$ and the second identity determines V_γ inductively. It follows from this construction that V is adapted. Therefore the space $\mathcal{G}_{\mathrm{AD}}$ of adapted $\Lambda\mathrm{GL}(\mathbb{C}^{N-1})$-valued functions is a subgroup of the gauge group which is called the adapted gauge group. The adapted gauge group acts on the space of families of connection 1-forms in the same way as same as the gauge group.

$$U^*\Omega^h = U^{-1}dU + U^{-1}\Omega^h U,$$

where $U \in \mathcal{G}_{\mathrm{AD}}$ and Ω^h is a family of connection 1-forms.

6.1. An adapted family of connection 1-forms

A family of connection 1-forms Ω^h is called adapted if Ω^h satisfies following properties.

(P) There is an $\mathrm{End}(\mathbb{C}^{N-1})$-valued polynomial $R = R^h(q)$ in q and h such that $\Omega^h = \frac{1}{h}R^h(q)\mathrm{d}t$.
(H) $\frac{1}{h}R^h(q)$ is homogeneous.
(I) $R^h(0) = I_{-1}$.
(N) The (-1)-diagonal component of $R^h(q)$ is I_{-1}. (normalization)

We denote by $\mathcal{A}_{\mathrm{AD}}$ the space of adapted families of connection 1-forms. Since $\partial = \partial/\partial t = q\partial/\partial q$ preserves degree, for $U \in \mathcal{G}_{\mathrm{AD}}$ and $\Omega^h \in \mathcal{A}_{\mathrm{AD}}$ the family of connection 1-forms $U^*\Omega^h$ is also adapted. In other words, the adapted gauge group $\mathcal{G}_{\mathrm{AD}}$ acts on the space $\mathcal{A}_{\mathrm{AD}}$ of adapted families of connection 1-forms.

Theorem 6.1 (*Uniqueness*). *If $\Omega_1^h, \Omega_2^h \in \mathcal{A}_{\mathrm{AD}}$ are $\frac{1}{h}$-linear (i.e. $h\Omega_1^h, h\Omega_2^h$ are independent of h) and adapted gauge equivalent, then $\Omega_1^h = \Omega_2^h$.*

Proof. Let $U \in \mathcal{G}_{\mathrm{AD}}$ such that $\Omega_2^h = U^*\Omega_1^h$. Since U is adapted, U can be written as

$$U = \tilde{Q}_0\left(I + h\tilde{Q}_1 + h^2\tilde{Q}_1 + \cdots + h^p\tilde{Q}_p\right)$$

for some integer p. Here $\tilde{Q}_i$ is independent of h. Note that $U|_{q=0} = I$ implies $\tilde{Q}_0|_{q=0} = I$ and $\tilde{Q}_i|_{q=0} = 0$. Comparing coefficients of powers of h in the identity

$$dU = U\Omega_2^h - \Omega_1^h U$$

gives the system

$$\begin{aligned}
&\Omega_1^h = \tilde{Q}_0\Omega_2^h\tilde{Q}_0^{-1},\\
&d\tilde{Q}_0 = \tilde{Q}_0[\tilde{Q}_1, h\Omega_2^h],\\
&d\tilde{Q}_\alpha = [\tilde{Q}_{\alpha+1}, h\Omega_2^h] - [\tilde{Q}_1, h\Omega_2^h]\tilde{Q}_\alpha \qquad (\alpha = 1, \ldots, p).
\end{aligned}$$

Here $\tilde{Q}_{p+1} = 0$. By the following Lemma 6.2, $\tilde{Q}_{\alpha+1} = 0$ implies that $\tilde{Q}_\alpha$ vanishes for $\alpha = 1, \ldots, p$ since $[\tilde{Q}_1, h\Omega_2^h]$ is homogeneous. $\tilde{Q}_1 = 0$ implies that $d\tilde{Q}_0$ also vanishes. Since $\tilde{Q}_0|_{q=0} = I$, we conclude that $\tilde{Q}_0 = I$. Thus $U = I$. □

Lemma 6.2. *Let $A = A(q)$ be a homogeneous* $\mathrm{End}(\mathbb{C}^{N-1})$*-valued polynomial in q such that $A(0) = 0$. If an* $\mathrm{End}(\mathbb{C}^{N-1})$*-valued polynomial $X = X(q)$ satisfies the following two conditions, then $X = 0$.*

(i) $dX = AX$.
(ii) *There is an integer m such that the n-diagonal component of X has degree $2n + 2m$ for each n.*

Proof. Let us write A and X as finite sums as follows:

$$A = \sum_{\alpha \geq 0} q^\alpha A_\alpha, \qquad X = \sum_{\alpha \geq 0} q^\alpha X_\alpha.$$

Note that the initial condition on A implies $A_0 = 0$. The equation can be written as

$$\gamma X_\gamma = \sum_{\alpha+\beta=\gamma} A_\alpha X_\beta \quad (\gamma = 0, 1, 2, \ldots).$$

We have $X_0 = 0$ since $0 = X_0 + A_0$. If $X_\beta = 0$ for $\beta = 0, \ldots, \gamma$, then the identity

$$(\gamma + 1)X_{\gamma+1} = \sum_{\beta=0}^{\gamma} A_{(\gamma+1)-\beta}X_\beta + A_0X_{\gamma+1}$$

implies $X_{\gamma+1} = 0$ because of $A_0 = 0$. By induction on γ, we conclude that $X_\gamma = 0$ for all γ, i.e., $X = 0$. □

6.2. An adapted system of ordinary differential equations

We consider the system $d\Phi = \Phi\Omega^h$ (with parameter h) of ordinary differential equations, where $\Phi = (\varphi_0, \ldots, \varphi_{N-2})$. The system $d\Phi = \Phi\Omega^h$ is called adapted if Ω^h is an adapted family of connection 1-forms.

If we introduce new unknown functions $\Psi = \Phi U$ with $U \in \mathcal{G}_{\mathrm{AD}}$, then the system of o.d.e. is equivalent to a new adapted system of o.d.e. $d\Psi = \Psi(U^*\Omega^h)$.

An adapted system can be reduced to an ordinary differential equation. There is an $\mathrm{End}(\mathbb{C}^{N-1})$-valued polynomial $R = R^h(q) = \left(r_{\alpha,\beta}\right)_{0\leq\alpha,\beta\leq N-2}$ in q and h such that $\Omega^h = \frac{1}{h}R^h(q)\mathrm{d}t$. Since $r_{\alpha,\beta} = 0$ for $\alpha > \beta + 1$, the system is written as

$$h\frac{\partial\varphi_\beta}{\partial t} = \sum_{\alpha=0}^{\beta+1} r_{\alpha,\beta}\varphi_\alpha \quad (\beta = 0, \ldots, N-2),$$

where $\varphi_{N-1} = 0$. Because $r_{\beta+1,\beta} = 1$ $(\beta = 0, \ldots, N-3)$,

$$\varphi_{\beta+1} = h\frac{\partial\varphi_\beta}{\partial t} - \sum_{\alpha=0}^{\beta} r_{\alpha,\beta}\varphi_\alpha \quad (\beta = 0, \ldots, N-2).$$

For each β, φ_β can be written in terms of φ_0 and its derivatives:

(P) There exist polynomials $\sigma_{\beta,\gamma}$ in q and h so that $\varphi_\beta = \sum\limits_{\gamma=0}^{\beta} \sigma_{\beta,\gamma} h^\gamma \frac{\partial^\gamma \varphi_0}{\partial t^\gamma}$.
(H) For each β and γ, $\sigma_{\beta,\gamma}$ is a homogeneous polynomial of degree $2(\beta-\gamma)$.
(I) If $\beta > \gamma$, then $\sigma_{\beta,\gamma}|_{q=0} = 0$.
(N) $\sigma_{\beta,\beta} = 1$.

If the above conditions hold for $\varphi_0, \ldots, \varphi_\beta$, then

$$\begin{aligned}\varphi_{\beta+1} &= h\frac{\partial}{\partial t}\left(\sum_{\gamma=0}^{\beta} \sigma_{\beta,\gamma} h^\gamma \frac{\partial^\gamma \varphi_0}{\partial t^\gamma}\right) - \sum_{\alpha=0}^{\beta} r_{\alpha,\beta} \sum_{\gamma=0}^{\alpha} \sigma_{\alpha,\gamma} h^\gamma \frac{\partial^\gamma \varphi_0}{\partial t^\gamma} \\ &= \sigma_{\beta,\beta} h^{\beta+1} \frac{\partial^{\beta+1}\varphi_0}{\partial t^{\beta+1}} + \sum_{\gamma=1}^{\beta}\left(\sigma_{\beta,\gamma-1} + qh\frac{\partial \sigma_{\beta,\gamma}}{\partial q} - \sum_{\alpha=\gamma}^{\beta} r_{\alpha,\beta}\sigma_{\alpha,\gamma}\right) h^\gamma \frac{\partial^\gamma \varphi_0}{\partial t^\gamma} \\ &\quad + \left(qh\frac{\partial \sigma_{\beta,0}}{\partial q} - \sum_{\alpha=0}^{\beta} r_{\alpha,\beta}\sigma_{\alpha,0}\right)\varphi_0.\end{aligned}$$

Since $\deg r_{\alpha,\beta} = 2(\beta-\alpha+1)$ and $\deg \sigma_{\beta,\gamma} = 2(\beta-\alpha+1)$, the conditions hold for $\varphi_{\beta+1}$.

In particular $\varphi_{N-1} = 0$. Therefore φ_0 satisfies the following o.d.e.:

$$\left((h\partial)^{N-1} + \sigma_{N-1,N-2}(h\partial)^{N-2} + \cdots + \sigma_{N-1,1}(h\partial) + \sigma_{N-1,0}\right)\varphi_0 = 0.$$

The above o.d.e is called the reduced equation of the system.

6.3. A $\mathcal{D}$-module and an adapted basis

Let $\Lambda = \mathbb{C}[q]$ and $\mathcal{D}$ be the module generated by $h\partial$ over $\Lambda(h)$. We consider a $\mathcal{D}$-module $\mathcal{M}^h = \mathcal{D}/(P)$ for some differential operator $P \in \mathcal{D}$. If the order of P is $N-1$, the rank of $\mathcal{M}^h$ is $N-1$ over $\Lambda(h)$. We assume that there exist homogeneous polynomials a_α of degree 2α such that $P = (h\partial)^{N-1} + \sum_{\alpha=1}^{N-1} a_\alpha (h\partial)^{N-1-\alpha}$. Let $P_0, \ldots, P_{N-2} \in \mathcal{D}$ be differential operators such that $[P_0], \ldots, [P_{N-2}]$ form a $\Lambda(h)$-basis of $\mathcal{M}^h$. We say that $[P_0], \ldots, [P_{N-2}]$ form an adapted ($\Lambda(h)$-)basis if the differential operators satisfy the following properties.

(P) There exist polynomials $c_{\beta,\gamma}$ in q and h so that $P_\beta = \sum_{\gamma=0}^{\beta} c_{\beta,\gamma}(h\partial)^\gamma$.
(H) For each β and γ, $c_{\beta,\gamma}$ is a homogeneous polynomial of degree $2(\beta-\gamma)$.
(I) If $\beta > \gamma$, then $c_{\beta,\gamma}|_{q=0} = 0$.
(N) $c_{\beta,\beta} = 1$.

For example, $[1], [h\partial] \ldots, [(h\partial)^{N-2}]$ form an adapted basis. A $\mathcal{D}$-module $\mathcal{M}^h = \mathcal{D}/(P)$ with an adapted basis $[P_0], \ldots, [P_{N-2}]$ defines a family of (flat) connection 1-forms Ω^h as in Section 5. Let $[P'_0], \ldots, [P'_{N-2}]$ be another adapted basis of $\mathcal{M}^h$. The adapted conditions imply that there exists an adapted gauge transformation $U \in \mathcal{G}_{\mathrm{AD}}$ such that $(P'_0, \ldots, P'_{N-2}) = (P_0, \ldots, P_{N-2})U$. The family Ω'^h of connection 1-forms associated with the adapted basis $[P'_0], \ldots, [P'_{N-2}]$ agrees with $U^*\Omega^h$. Conversely, an adapted family Ω^h of connection 1-forms defines a $\mathcal{D}$-module $\mathcal{M}^h$ and an adapted $\Lambda(h)$-basis as follows. There is an $\mathrm{End}(\mathbb{C}^{N-1})$-valued polynomial $R = R^h(q) = \left(r_{\alpha,\beta}\right)_{0\le\alpha,\beta\le N-2}$ in q and h such that $\Omega^h = \frac{1}{h}R^h(q)\mathrm{d}t$. The differential operators $P_0, \ldots, P_{N-1}$ are defined inductively as follows.

$$P_0 = 1, \qquad P_{\beta+1} = (h\partial)P_\beta - \sum_{\alpha=0}^{\beta} r_{\alpha,\beta}P_\alpha \quad (\beta = 0, \ldots, N-2).$$

We define a $\mathcal{D}$-module $\mathcal{M}^h$ by $\mathcal{D}/(P_{N-1})$ and we call the operator P_{N-1} the reduced operator of Ω^h. As in Section 6.2, $[P_0], \ldots, [P_{N-2}]$ form an adapted $\Lambda(h)$-basis of $\mathcal{M}^h$. By the definition of the operators $P_0, \ldots, P_{N-2}$, we have $h\partial([P_0], \ldots, [P_{N-2}]) = ([P_0], \ldots, [P_{N-2}])R^h(q)$.

Lemma 6.3. *Let $\mathcal{M}^h$ be a $\mathcal{D}$-module $\mathcal{D}/(P)$ of rank $N-1$ and Ω^h an adapted family associated with an adapted basis $[P_0], \dots, [P_{N-2}]$. Then the reduced operator of Ω^h agrees with P. In particular, the reduced operator is independent of the choice of the adapted basis.*

Proof. Let $P' = (h\partial)P_{N-2} - \sum_{\alpha=0}^{N-2} r_{\alpha,N-2}(h\partial)^\alpha$ be the reduced operator of Ω^h. By the definition of $\Omega^h = \frac{1}{h}\left(r_{\alpha,\beta}\right)_{0\le\alpha,\beta\le N-2} dt$, we have

$$\left[(h\partial)P_{N-2}\right] = \left[\sum_{\alpha=0}^{N-2} r_{\alpha,N-2}(h\partial)^\alpha\right], \quad \text{i.e. } [P'] = 0.$$

Since the monic polynomial $P' \in \Lambda(h)[h\partial]$ has order $N-1$, P' must agree with P. □

Theorem 6.4 (*Equivalence*). *Let $\Omega_1^h, \Omega_2^h \in \mathcal{A}_{\mathrm{AD}}$. Then Ω_1^h and Ω_2^h have the same reduced operator if and only if Ω_1^h and Ω_2^h are adapted gauge equivalent.*

Proof. First, we assume that Ω_1^h and Ω_2^h have the same reduced operator. Let $P_0^\alpha, \dots, P_{N-1}^\alpha$ be the operators associated with Ω_α^h for $\alpha = 1, 2$. By the assumption, $P = P_{N-1}^1 = P_{N-1}^2$. Since two bases $[P_0^1], \dots, [P_{N-2}^1]$ and $[P_0^2], \dots, [P_{N-2}^2]$ are adapted bases of the same $\mathcal{D}$-module $\mathcal{M}^h = \mathcal{D}/(P)$, there exists an adapted gauge transformation $U \in \mathcal{G}_{\mathrm{AD}}$ such that $([P_0^2], \dots, [P_{N-2}^2]) = ([P_0^1], \dots, [P_{N-2}^1])U$. If $R_\alpha^h(q)$ is a matrix-valued polynomial in q and h such that $\Omega_\alpha^h = \frac{1}{h}R_\alpha^h(q)$, then $h\partial([P_0^\alpha], \dots, [P_{N-2}^\alpha]) = ([P_0^\alpha], \dots, [P_{N-2}^\alpha])R_\alpha^h(q)$ for $\alpha = 1, 2$. Therefore we conclude that $\Omega_2^h = U^*\Omega_1^h$.

Next, we assume that there exists an adapted gauge transformation $U \in \mathcal{G}_{\mathrm{AD}}$ such that Ω_1^h and Ω_2^h. Let $\mathcal{M}^h$ with $[P_0^1], \dots, [P_{N-2}^1]$ be the $\mathcal{D}$-module with the adapted basis defined by Ω_1^h and P_{N-1}^1 the reduced operator of Ω_1^h. Define an adapted basis $[P_0^2], \dots, [P_{N-2}^2]$ by $(P_0^2, \dots, P_{N-2}^2) = (P_0^1, \dots, P_{N-2}^1)U$. Then the adapted family Ω_2^h agrees with the adapted family defined by the adapted basis $[P_0^2], \dots, [P_{N-2}^2]$ of $\mathcal{M}^h = \mathcal{D}/(P_{N-1}^1)$. According to Lemma 6.3, the reduced operator of Ω_2^h agrees with P_{N-1}^1. □

7. Relation between Birkhoff factorization and Jinzenji's results

Recall the quantum differential system (with parameter h) for M_N^k:

$$h\frac{\partial\psi_{N-2-m}}{\partial t} = \psi_{N-1-m}(t) + \sum_{d\ge1} L_m^d q^d \psi_{N-1-m-(N-k)d}(t),$$
$$h\frac{\partial\psi_{N-2}}{\partial t} = \sum_{d\ge1} L_0^d q^d \psi_{N-1-(N-k)d}(t),$$

(where $m = 1, \dots, N-2$). Note that $\deg h = 2$ and $\deg q = 2(N-k)$. We can write the above system of o.d.e. with the restricted Dubrovin connection $\Omega_{\mathrm{D}}^h \in \mathcal{A}_{\mathrm{AD}}$:

$$d\Psi = \Psi\Omega_{\mathrm{D}}^h, \quad \text{where } \Psi = (\psi_0, \dots, \psi_{N-2}).$$

Theorem 4.1 says that the reduced operator of Ω_{D}^h at $h = 1$ agrees with the Picard–Fuchs operator $\partial^{N-1} - kq(k\partial + (k-1)) \cdots (k\partial + 1)$ if $N - k \ge 2$.

Theorem 7.1. *Let $\Omega^h \in \mathcal{A}_{\mathrm{AD}}$ be an adapted family of connection 1-forms whose reduced operator agrees with $P^{N,k} = (h\partial)^{N-1} - kqh^{k-1}(k\partial + (k-1)) \dots (k\partial + 1)$. If Ω^h is h^{-1}-linear, then $\Omega^h = \hat{\Omega}^h$.*

Note that we have the adapted family $\Omega_{\mathrm{PF}}^h \in \mathcal{A}_{\mathrm{AD}}$, whose reduced operator agrees with $P^{N,k}$. Using a matrix-valued function L which satisfies $\Omega_{\mathrm{PF}}^h = L^{-1}dL$, $\hat{\Omega}^h$ is defined as $(L_-)^{-1}dL_-$ in Section 5. Here L_- is the first factor of the Birkhoff factorization of $L = L_-L_+$. Since $L_+ \in \mathcal{G}_{\mathrm{AD}}$, Ω_{PF}^h and $\hat{\Omega}^h$ are adapted gauge equivalent.

Proof. Note that $\hat{\Omega}^h$ is adapted and h^{-1}-linear. According to Theorem 6.4, Ω^h_{PF} and $\hat{\Omega}^h$ has the same reduced operator $P^{N,k}$, because Ω^h_{PF} and $\hat{\Omega}^h$ are adapted gauge equivalent. Therefore $\hat{\Omega}^h$ satisfies the conditions of the theorem.

If Ω^h satisfies the conditions of the theorem, then Theorem 6.4 says that Ω^h and $\hat{\Omega}^h$ are adapted gauge equivalent. Moreover the fact that Ω^h and $\hat{\Omega}^h$ are h^{-1}-linear implies $\Omega^h = \hat{\Omega}^h$ because of Theorem 6.1. □

However the reduced operator of the quantum differential system for M_N^k differs for the cases $N-k \geq 2$ and $N-k=1$; Jinzenji considered an adapted and h^{-1}-linear family $\Omega^h_{\mathrm{J}} \in \mathcal{A}_{\mathrm{AD}}$ whose reduced operator agrees with $P^{N,k}$ and he named the coefficients of Ω^h_{J} the virtual structural constants. Moreover using Beauville's result [2] for the case of $N-2k \geq 0$ as the initial data, he gave explicit formula for Ω^h_{J} in [8]. Jinzenji's explicit formulae guarantee the existence of Ω^h_{J}. Since the adapted family Ω^h_{J} automatically satisfies the conditions of the above theorem, Ω^h_{J} agrees with $\hat{\Omega}^h$.

Corollary 7.2. *The adapted family Ω^h_{J} agrees with $\hat{\Omega}^h$.*

In the case $N-k \geq 2$, Ω^h_{J} a priori agrees with the restricted Dubrovin connection 1-form Ω^h_{D}, and hence so does $\hat{\Omega}^h$.

In the case $N-k=1$, the Dubrovin connection 1-form has a reduced operator different from that for the case of $N-k \geq 2$.

Theorem 7.3 (*Givental*). *Let $S = \exp\left(-\frac{(N-1)!q}{h}\right) I$. The quantum differential system for M_N^{N-1} can be written as*

$$d\Psi = \Psi(S^* \Omega^h_{\mathrm{J}}).$$

Note that S is not adapted. The above theorem implies

$$\Omega^h_{\mathrm{D}} = S^* \Omega^h_{\mathrm{J}} = S^* \hat{\Omega}^h = -\frac{(N-1)!q}{h} I\,dt + \hat{\Omega}^h.$$

Acknowledgment

I would like to express my heartfelt thanks to Martin Guest, my adviser, for his guidance and insight.

References

[1] A. Amarzaya, M.A. Guest, Gromov–Witten invariants of flag manifolds, via D-modules, J. London Math. Soc. (2) 72 (1) (2005) 121–136.
[2] Arnaud Beauville, Quantum cohomology of complete intersections, Mat. Fiz. Anal. Geom. 2 (3–4) (1995) 384–398.
[3] Aaron Bertram, Holger P. Kley, New recursions for genus-zero Gromov–Witten invariants, Topology 44 (1) (2005) 1–24.
[4] Alberto Collino, Masao Jinzenji, On the structure of the small quantum cohomology rings of projective hypersurfaces, Comm. Math. Phys. 206 (1) (1999) 157–183.
[5] Kenji Fukaya, Kaoru Ono, Arnold conjecture and Gromov–Witten invariant, Topology 38 (5) (1999) 933–1048.
[6] Alexander B. Givental, Equivariant Gromov–Witten invariants, Int. Math. Res. Not. (13) (1996) 613–663.
[7] Martin A. Guest, Quantum cohomology via D-modules, Topology 44 (2) (2005) 263–281.
[8] Masao Jinzenji, Gauss–Manin system and the virtual structure constants, Int. J. Math. 13 (5) (2002) 445–477.
[9] Jun Li, Gang Tian, Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds, First Int. Press Lect. Ser., I (1998) 47–83.
[10] Rahul Pandharipande, Rational curves on hypersurfaces (after A. Givental), in: Séminaire Bourbaki., vol. 1997–98, 1998, pp. 307–340. (Astérisque, No. 252, Exp. No. 848, 5).
[11] Yongbin Ruan, Virtual neighborhoods and pseudo-holomorphic curves, Turkish J. Math. 23 (1) (1999) 161–231.